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FRICTIONLESS CONTACT PROBLEM FOR AN ELASTIC LAYER UNDER AXISYMMETRIC LOADING†

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Abstract—The paper considers the frictionless contact problem for an elastic layer which is resting on a rigid horizontal foundation and is acted upon by an axisymmetric line load P, a uniform clamping pressure p_0, and a vertical homogeneous body force ρg, due to gravity. For varying values of the radius of the loading ring and of the load ratio $P/(\rho gh + p_0)h$, the critical value of the applied load initiating an interface separation along the contact plane, the size of the separation area, and the distribution of the contact stress are studied, and some numerical results are given.

1. INTRODUCTION

Prior to the publication of [1], in solving the problems involving elastic layers and foundations it was generally assumed that the contact between the layer and the subspace is either one of perfect adhesion or frictionless with the additional condition that across the interface the normal component of the displacement vectors is continuous, i.e. on the interface no separation is allowed (see for example [2] for some typical results and references). However, in [1] it was pointed out that if the layer is loaded by local compressive forces, due to the "bending" effects the contact area would decrease, and the size of the resulting contact area would be independent of the magnitude of the load and would depend on its relative distribution only. This peculiar property of the so-called receding contact problem holds also for loading through a flat-ended rigid stamp with sharp edges [3, 4]. For other stamp profiles the size of the contact area becomes a function of the resultant compressive force [3, 4]. Clearly, for a locally loaded frictionless infinite layer or plate, formation of a finite contact area is possible only if one neglects the effect of gravity (see for example [5–9]). Some plane contact problems for a frictionless layer or a beam resting on a horizontal rigid foundation were considered in [10–12] where the effect of gravity (and a possible uniform clamping pressure) was taken into account. In such problems as long as the external load is applied locally, the contact area would always be infinite, and when the magnitude of the external load exceeds a certain critical value a separation would take place between the layer and the foundation.

The purpose of this paper is to study this separation process for an infinite elastic layer under axisymmetric loading. Specifically, the contact pressure, the critical load initiating separation, and the size of the separation area will be investigated and determined as a function of some dimensionless load and geometrical variables.

2. FORMULATION

The axisymmetric elasticity problem is described in Fig. 1. If the magnitude of the load P is less than a critical value P_{cr}, there is no separation along the interface and the problem can be solved as an ordinary boundary value problem (Fig. 1a). However, for $P > P_{cr}$ the separation occurs, and depending on r_0/h one would have the mixed boundary value problem shown in either Fig. 1(b) or Fig. 1(c). In either case the following equations must be solved under certain boundary conditions:

$$(\lambda + 2\mu)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 w}{\partial r \partial z}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial r \partial z}\right) = 0,$$

$$(\lambda + 2\mu)\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\mu}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) - \mu\left(\frac{\partial^2 u}{\partial r \partial z} - \frac{\partial^2 w}{\partial r^2}\right) = \rho g, \qquad \text{(1a, b)}$$

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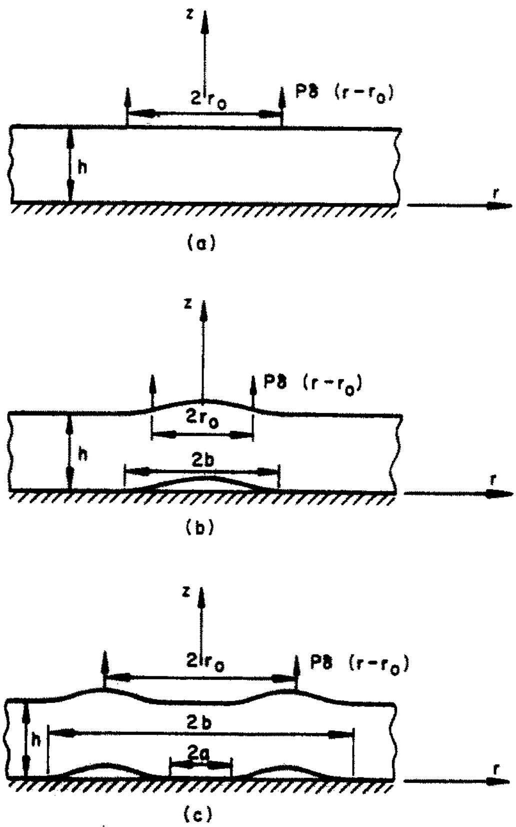

Fig. 1. Geometry of the frictionless contact problem with and without interface separation.

where u and w are the r and z-components of the displacement and ρg is the body force due to gravity. The stress components are given by

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right),$$

$$\sigma_{\theta\theta} = (\lambda + 2\mu)\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right),$$

$$\sigma_{zz} = (\lambda + 2\mu)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right),$$

$$\sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right). \tag{2a–d}$$

Writing the solution of (1) as

$$u(r, z) = u_p(r) + u_h(r, z)$$

$$w(r, z) = w_p(z) + w_h(r, z), \tag{3a,b}$$

the particular solution corresponding to the nonhomgeneous term ρg in (1) and satisfying

$$w_p(0) = 0, \qquad \sigma^p_{zz}(r, 0) = -p_e, \qquad \sigma^p_{zz}(r, h) = -p_0,$$

$$\int_0^h \sigma^p_{rr}(r, z)\,\mathrm{d}z = 0, \qquad (p_e = p_0 + \rho g h), \tag{4a–d}$$

may be obtained as follows:

$$u_p(r)=\frac{1}{2\mu}\frac{3-\kappa}{7-\kappa}\left(p_0+\frac{1}{2}\rho gh\right)r,$$

$$w_p(z)=\frac{2p_0z}{\mu(\kappa-7)}+\frac{\rho gz}{2\mu}\left[\frac{\kappa-1}{\kappa+1}(z-h)+\frac{2h}{\kappa-7}\right], \tag{5a,b}$$

where p_0 is the uniform clamping pressure applied to the layer at $z=h$ and $\kappa=3-4\nu$. The homogeneous solution of (1) satisfying the regularity condition at $r=\infty$ may be expressed as [4]

$$u_h(r,z)=\int_0^\infty[(A_1+A_2z)\,e^{-\alpha z}+(A_3+A_4z)\,e^{\alpha z}]J_1(\alpha r)\alpha\,d\alpha,$$

$$w_h(r,z)=\int_0^\infty\left\{\left[A_1+\left(\frac{\kappa}{\alpha}+z\right)A_2\right]e^{-\alpha z}+\left[-A_3+\left(\frac{\kappa}{\alpha}-z\right)A_4\right]e^{\alpha z}\right\}J_0(\alpha r)\alpha\,d\alpha,$$

$$0<z<h,\qquad 0\le r<\infty. \tag{6a,b}$$

Substituting (3), (5) and (6) into (2), the stress components of interest are found to be:

$$\frac{1}{2\mu}\sigma_{zz}(r,z)=\int_0^\infty\left\{-\left[\alpha\left(A_1+A_2z\right)+\frac{1+\kappa}{2}A_2\right]e^{-\alpha z}+\left[-(A_3+A_4z)+\frac{1+\kappa}{2}A_4\right]e^{\alpha z}\right\}$$
$$\times J_0(\alpha r)\alpha\,d\alpha+\frac{1}{2\mu}[\rho g(z-h)-p_0],$$

$$\frac{1}{2\mu}\sigma_{rz}(r,z)=\int_0^\infty\left\{-\left[\alpha(A_1+A_2z)-\frac{1-\kappa}{2}A_2\right]e^{-\alpha z}+\left[\alpha(A_3+A_4z)+\frac{1-\kappa}{2}A_4\right]e^{\alpha z}\right\}$$
$$\times J_1(\alpha r)\alpha\,d\alpha. \tag{7a,b}$$

The unknown functions $A_1(\alpha)\ldots A_4(\alpha)$ are determined from the boundary conditions at $z=0$ and $z=h$.

3. CONTINUOUS CONTACT ($0<P<P_{cr}$)

Referring to Fig. 1a let P be the magnitude of the tensile line load (per unit length) acting along the circle $r=r_0$, $z=h$. If P is sufficiently small so that no separation takes place on the interface $z=0$ (i.e. for $P<P_{cr}$), the solution of the problem as given by (3), (5) and (6) may be completed by using eqns (7) and determing $A_1,\ldots,A_4$ from the following boundary conditions:

$$\sigma_{rz}(r,0)=0,\qquad w(r,0)=0,$$

$$\sigma_{rz}(r,h)=0,\qquad \sigma_{zz}(r,h)=P\delta(r-r_0)-p_0,\qquad 0\le r<\infty. \tag{8a–d}$$

After determining A_i, the contact stress is found to be

$$\sigma_{zz}(r,0)=-p_0-\rho gh+2Pr_0\int_0^\infty\frac{(\alpha h+1)\,e^{-\alpha h}+(\alpha h-1)\,e^{-3\alpha h}}{1+4\alpha h\,e^{-2\alpha h}-e^{-4\alpha h}}J_0(\alpha r_0)J_0(\alpha r)\alpha\,d\alpha. \tag{9}$$

Defining now the following quantities

$$\omega=\alpha h,\qquad \sigma_{zz}(r,0)=p(r),\qquad p_0+\rho gh=p_e,\qquad \gamma=\frac{P}{hp_e}, \tag{10}$$

eqn (9) may be written as

$$\frac{p(r)}{p_e}=-1+2\frac{r_0}{h}\gamma\int_0^\infty\frac{(\omega+1)\,e^{-\omega}+(\omega-1)\,e^{-3\omega}}{1+4\omega\,e^{-2\omega}-e^{-4\omega}}J_0\left(\omega\frac{r_0}{h}\right)J_0\left(\omega\frac{r}{h}\right)\omega\,d\omega,\qquad 0\le r<\infty. \tag{11}$$

From (11) it is seen that for $\gamma = 0$, $p(r) = -p_e$ and up to a certain value of γ, $p(r)$ would remain to be negative, that is, the continuous contact on $z = 0$ plane would be maintained. Also note that this critical value $\gamma_{cr} = P_{cr}/p_e h$, at which the separation on $z = 0$ starts is a function of r_0/h. Clearly, for small values of r_0/h the separation starts at $r = 0$ and γ_{cr} may be obtained from

$$\frac{1}{\gamma_{cr}} = 2\frac{r_0}{h}\int_0^\infty \frac{(\omega+1)\,e^{-\omega} + (\omega-1)\,e^{-3\omega}}{1 + 4\omega\,e^{-2\omega} - e^{-4\omega}} J_0\left(\omega\frac{r_0}{h}\right)\omega\,d\omega, \qquad \left(0 \le \frac{r_0}{h} < 0.62\right). \tag{12}$$

The calculated results show that (12) would give γ_{cr} if $0 \le r_0/h < 0.62$. For values of r_0/h greater than 0.62 the equation obtained from (11) by writing $p(r) = 0$ has two unknowns, γ_{cr} and r/h. For some selected values of r_0/h Fig. 2 shows the pressure distribution, the value of γ_{cr} and the corresponding r/h at which $p(r) = 0$. The calculated values of γ_{cr} are shown in Fig. 3 as a function of r_0/h. Since the critical value of the resultant force $P_R = 2\pi r_0 P_{cr}$ would remain finite as $r_0 \to 0$, for $r_0 = 0$ P_{cr} as well as γ_{cr} would become unbounded. Therefore, for small values of

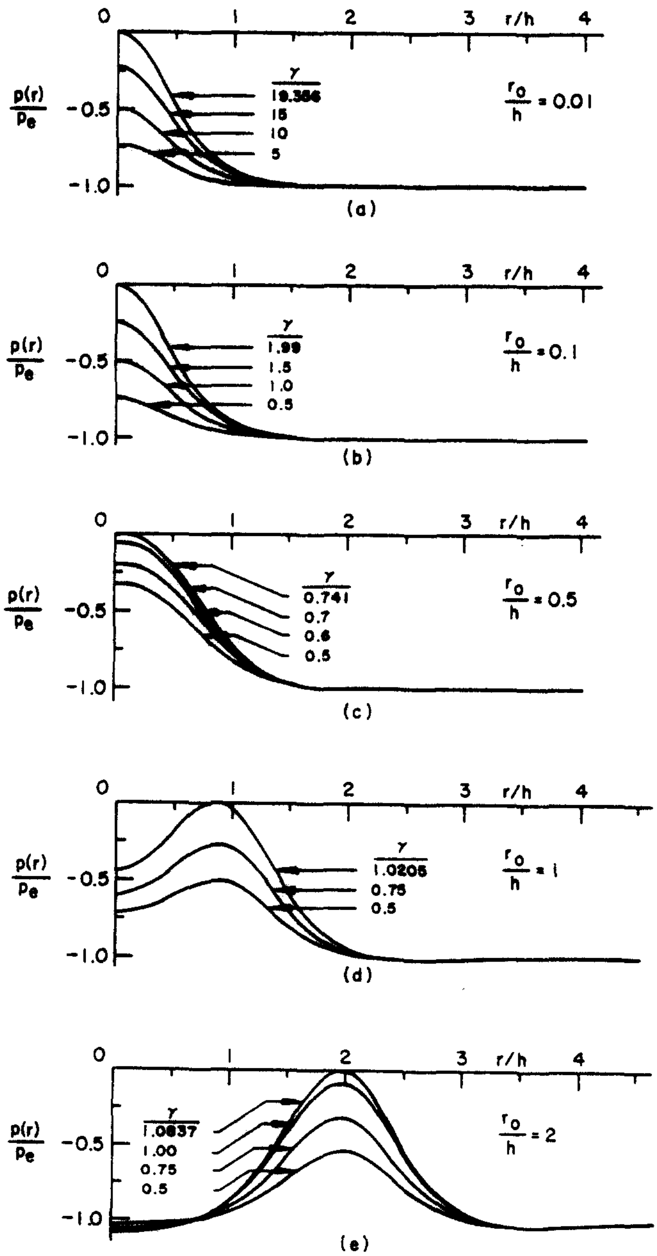

Fig. 2. Contact stress distribution in the absence of interface separation.

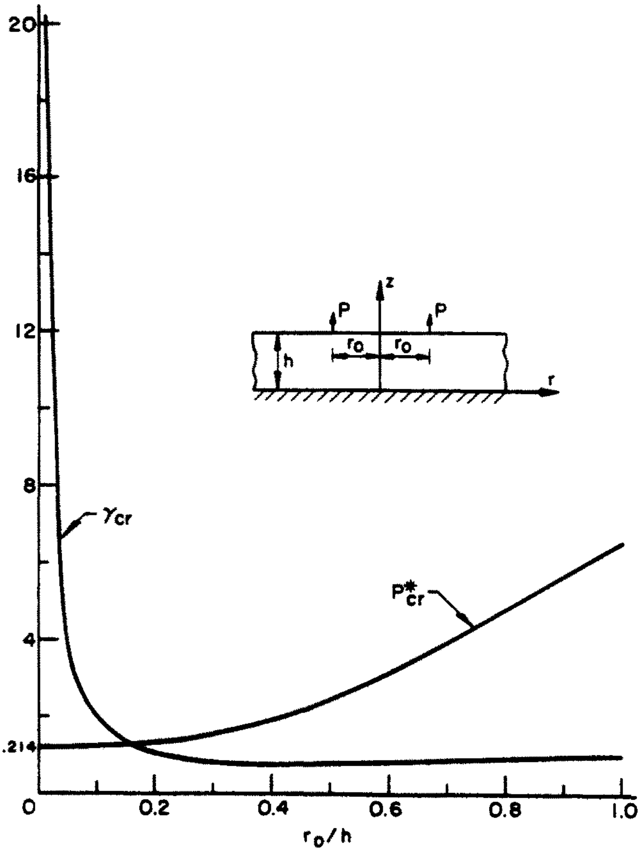

Fig. 3. Critical load initiating interface separation. $\gamma_{cr} = P_{cr}/h(\rho gh + p_0)$, $P^* = 2\pi r_0 P_{cr}/h(\rho gh + p_0)$.

r_0/h a more appropriate dimensionless quantity which should be evaluated would be

$$P^*_{cr} = \frac{P_R}{p_e h^2} = \frac{2\pi r_0}{p_e h^2} P_{cr} = \frac{2\pi r_0}{h} \gamma_{cr}. \tag{13}$$

Figure 3 also shows P^*_{cr} as a function of r_0/h. Note that for a concentrated lifting load (i.e. for $r_0 = 0$) the (total) force initiating an interface separation is given by

$$P_R = 1.214\,(p_0 + \rho gh)h^2. \tag{14}$$

4. SEPARATION ($P > P_{cr}$)

For a given value of r_0/h if $P > P_{cr}$ there will be separation on the $z = 0$ plane, generally along a ring-shaped region $a < r < b$ (Fig. 1c). For this problem eqns (3), (5)–(7) are still valid. However, in this case, the functions $A_1, \ldots A_4$ must be determined from the following mixed boundary conditions.

$$\sigma_{zz}(r, h) = P\delta(r - r_0) - p_0, \quad \sigma_{rz}(r, h) = 0, \quad 0 \le r < \infty, \tag{15a,b}$$

$$\sigma_{rz}(r, 0) = 0, \quad 0 \le r < \infty, \tag{15c}$$

$$\sigma_{zz}(r, 0) = 0, \quad a < r < b, \tag{15d}$$

$$w(r, 0) = 0, \quad 0 \le r < a, \quad b < r < \infty. \tag{15e}$$

Three of the four unknowns A_i may be eliminated by substituting from (7) into (15a–c) and the mixed conditions (15d) and (15e) would give a pair of dual integral equations for the remaining unknown function. One may also reduce the problem to a singular integral equation by defining the following function

$$\frac{\partial}{\partial r} w(r,0) = f(r), \qquad 0 \le r < \infty. \tag{16}$$

Replacing now the conditions (15d) and (15e) by (16), $A_1, \ldots A_4$ may be determined in terms of $f(r)$, and (15d) would then give an integral equation to determine $f(r)$. Note that (15e) is equivalent to

$$f(r) = 0, \qquad 0 \le r < a, \qquad b < r < \infty, \qquad \int_a^b f(r)\,dr = 0. \tag{17a,b}$$

After somewhat lengthy but straightforward manipulations we obtain (see for example Ref. [4] for the general procedure and for the derivation of singular kernel)

$$\sigma_{zz}(r,0) = \frac{4\mu}{1+\kappa}\left\{\frac{1}{\pi}\int_a^b \frac{h_1(r,s)}{s-r} f(s)\,ds + \int_a^b f(s)\,ds \int_0^\infty E_1(\alpha)J_0(\alpha r)J_1(\alpha s)\alpha\,d\alpha\right\}$$
$$+ P r_0 \int_0^\infty E_2(\alpha)J_0(\alpha r_0)J_0(\alpha r)\alpha\,d\alpha - p_e = 0, \qquad a < r < b, \tag{18}$$

where

$$h_1(r,s) = \begin{cases} \dfrac{2r}{s+r}\left[\dfrac{s^2-r^2}{r^2} K(s/r) + E(s/r)\right], & s < r, \\ \dfrac{2s}{s+r} E(r/s), & r < s, \end{cases}$$

$$E_1(\alpha) = \frac{e^{-4\alpha h} - (2\alpha^2 h^2 + 2\alpha h + 1)\,e^{-2\alpha h}}{2(1 + 4\alpha h\, e^{-2\alpha h} - e^{-4\alpha h})},$$

$$E_2(\alpha) = \frac{(\alpha h + 1)\,e^{-\alpha h} + (\alpha h - 1)\,e^{-3\alpha h}}{2(1 + 4\alpha h\, e^{-2\alpha h} - e^{-4\alpha h})}. \tag{19a–c}$$

In (19a) K and E are respectively the complete elliptic integrals of the first and the second kind. For $0 < a < b$, the singular integral equation (18) must be solved under the single-valuedness condition (17b). If r_0 is relatively small so that $a = 0$ and the separation area is circular (Fig. 1b), the integral equation (18) remains the same except for the dominant part (i.e. the first term on the l.h.s.) which becomes

$$\frac{4\mu}{1+\kappa}\frac{1}{\pi}\int_0^a h_2(r,s)\left(\frac{1}{s-r} - \frac{1}{s+r}\right) f(s)\,ds, \tag{20}$$

where

$$h_2(r,s) = \begin{cases} \dfrac{s^2-r^2}{r^2} K(s/r) + E(s/r), & s < r, \\ \dfrac{s}{r} E(r/s), & s > r. \end{cases} \tag{21}$$

In this case the condition (17b) is not valid and also is not required for a unique solution of the integral equation. It should be noted that in both cases the solution is obtained within an arbitrary rigid body displacement in w which can be chosen as being zero.

For a given r_0/h and $\gamma > \gamma_{cr}$ if $a > 0$ then both a and b are unknown constants. Also we note that because of the requirements of smooth contact at a and b, $f(a) = 0 = f(b)$ and therefore the index of the singular integral equation (18) is (-1)[13]. Thus, in solving the problem the two conditions which would account for the unknowns a and b are the condition (17b) and the consistency condition of the integral equation[13]. To solve the problem numerically we first define the following new variables and quantities:

$$
\begin{aligned}
x &= [2r-(b+a)]/(b-a),\\
t &= [2s-(b+a)]/(b-a),\\
\omega &= h\alpha,\\
\phi(t) &= \frac{4\mu}{(\kappa+1)p_e} f(r),\\
k(x,t) &= \frac{1}{t-x}[h_1(r,s)-1],\\
k_1(x,t) &= \frac{b-a}{2h}\left(\frac{b-a}{2h}t+\frac{b+a}{2h}\right)\int_0^{\infty} E_1\left(\frac{\omega}{h}\right) J_0(\alpha r) J_1(\alpha s)\omega\, d\omega,\\
k_2(x) &= \frac{r_0}{h}\int_0^{\infty} E_2\left(\frac{\omega}{h}\right) J_0\left(\frac{r_0\omega}{h}\right) J_0(r\alpha)\omega\, d\omega.
\end{aligned}
\tag{22}
$$

The integral equation (18) and condition (17b) may then be expressed as

$$\frac{1}{\pi}\int_1^1 \frac{\phi(t)}{t-x}\,dt + \int_{-1}^1\left[\frac{1}{\pi}k(x,t)+k_1(x,t)\right]\phi(t)\,dt = 1-\gamma k_2(x), \qquad -1<x<1, \tag{23}$$

$$\int_{-1}^1 \phi(t)\,dt = 0. \tag{24}$$

Writing the solution of (23) in the form

$$\phi(t) = G(t)(1-t^2)^{1/2}, \qquad -1<t<1 \tag{25}$$

eqns (23) and (24) may be reduced to [14]

$$\sum_{i=1}^{n}\frac{1-t_i^2}{n+1}G(t_i)\left[\frac{1}{t_i-x_j}+k(x_j,t_i)+\pi k_1(x_j,t_i)\right] = 1-\gamma k_2(x_j), \qquad (j=1,\ldots n+1) \tag{26}$$

$$\sum_1^n \frac{1-t_i^2}{n+1}G(t_i) = 0, \tag{27}$$

$$t_i = \cos\left(\frac{i\pi}{n+1}\right), \qquad n=1,\ldots n, \tag{28}$$

$$x_j = \cos\left[\frac{\pi}{2}\left(\frac{2j-1}{n+1}\right)\right], \qquad j=1,\ldots n+1. \tag{29}$$

In [14] it was shown that the extra equation in (26) corresponds to the consistency condition of the original integral equation (23). Thus the $(n+2)$ equations given by (26) and (27) determine the $(n+2)$ unknowns $G(t_i)$, $(i=1,\ldots n)$, a and b.

Noting that (18) gives $\sigma_{zz}(r,0)$ on the entire $z=0$ plane (i.e for $0\le r<\infty$), after determining $\phi(t)$ the contact stress outside the separation region (i.e. in $0\le r<a$, $b<r<\infty$) may be obtained from

$$\frac{1}{p_e}\sigma_{zz}(r,0) = \frac{p(r)}{p_e} = \frac{1}{p_e}p\left(\frac{b-a}{2}x+\frac{b+a}{2}\right) = \frac{1}{\pi}\int_{-1}^1 \frac{\phi(t)}{t-x}\,dt + \int_{-1}^1$$

$$\times\left[\frac{1}{\pi}k(x,t)+k_1(x,t)\right]\phi(t)\,dt - 1 + \gamma k_2(x), \qquad -\frac{b+a}{b-a}<x<-1, \qquad 1<x<\infty. \tag{30}$$

In the case of circular separation region (i.e. for $a = 0$), we also have $f(0) = 0 = f(b)$ and only b is unknown. Therefore, the index of the singular integral equation is again -1, and the solution must satisfy the consistency condition which accounts for the unknown constant b.

5. THE PLATE ASSUMPTION

A simple approximation to the lifting problem may be obtained by using a standard plate theory to determine the displacements in the elastic layer. For example assuming that r_0/h is sufficiently small and $a = 0$, by using the classical plate theory the problems can be formulated as follows:

$$\frac{D}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} = p_e - P\delta(r - r_0), \qquad r < b, \tag{31}$$

$$D = \frac{8\mu}{1+\kappa}\frac{h^3}{12}, \qquad w(b) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}r}w(b) = 0, \qquad \frac{\mathrm{d}^3}{\mathrm{d}r^3}w(0) = 0, \qquad w(0) < \infty, \tag{32a–d}$$

giving

$$w(r) = \frac{p_e(b^2 - r^2)^2}{64D} - \frac{Pr_0}{8D}\{b^2 - r^2[1 + 2\log b - 2H(r - r_0)\log r]\}, \qquad 0 \le r < b. \tag{33}$$

In (33) the constant b is the only unknown which needs to be determined. On the other hand, physically the problem has two more boundary conditions which have not yet been satisfied, namely the conditions of zero bending moment and zero transverse shear at $r = b$. It is thus clear that the classical plate theory cannot provide a solution which satisfies all the physical boundary conditions. If b is determined by using the condition that at $r = b$ the bending moment is zero (i.e. $w''(b) = 0$), we find

$$b^2 = \frac{4r_0 P}{p_e}, \tag{34}$$

or

$$\frac{b}{h} = 2(\gamma r_0/h)^{1/2}. \tag{35}$$

In this case the force equilibrium requires that a transverse shear $V = r_0P/b$ be applied along the circle $r = b$ to the plate, giving the contact stress as

$$\sigma_{zz}(r,0) = \begin{cases} -\left[p_e + \frac{r_0 P}{b}\delta(r - b)\right], & b \le r < \infty, \\ 0, \quad 0 \le r < b. \end{cases} \tag{36}$$

A less physically acceptable solution from the plate theory may be obtained by determining b from the condition of zero transverse shear at $r = b$, which requires that a concentrated bending moment be applied to the plate along the circle $r = b$ to maintain its equilibrium.

6. THE CASE OF COMPRESSIVE FORCE

The formulation of the contact problem given in Sections 2–4 is applicable to the case of compressive force (see the insert in Figs. 9–12) as well as the lifting force shown in Fig. 1. In the case of compressive force, P and $\gamma = P/(hp_e)$ are negative quantities. For values of P satisfying $p(r) < 0$, $0 \le r < \infty$, (11) is still valid and gives the contact pressure. For a given r_0, at a critical value $P = P_{cr}$ or $\gamma = \gamma_{cr}$ contact stress becomes zero at $r = r_{cr}$ where both γ_{cr} and r_{cr}

are unknown. These unknowns are again determined from

$$p(r_{cr}) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}r}p(r_{cr}) = 0, \tag{37}$$

where $p(r)$ is given by (11).

If the magnitude of P exceeds P_{cr} (i.e. for $|P| > |P_{cr}|$), then there will be separation on the contact plane $z = 0$. For relatively small values of r_0 the separation would take place along a ring-shaped region $a < r < b$ where $a > r_0$. In this case the integral equation (18) and the single-valuedness condition (17b) are still valid. However, for large values of r_0 if one continues to increase the magnitude of P beyond P_{cr}, at a certain value $|P| = |P'_{cr}|$ the contact pressure becomes zero at $r = 0$ and for $|P| > |P'_{cr}|$ one would have two separation zones, $0 \le r < c$ and $a < r < b$, where $0 < c < r_0 < a < b$. In this case the integral equations of the problem and the single-valuedness condition become

$$\frac{4\mu}{1+\kappa}\frac{1}{\pi}\left\{\int_0^c \left[\left(\frac{1}{s-r} - \frac{1}{s+r}\right)h_2(r,s) + h_3(r,s)\right]f_1(s)\,\mathrm{d}s \right.$$
$$\left. + \int_a^b \left[\frac{1}{s-r}h_1(r,s) + h_3(r,s)\right]f_2(s)\,\mathrm{d}s\right\} = p_e - Ph_4(r), \qquad 0 \le r < c,$$
$$\frac{4\mu}{1+\kappa}\frac{1}{\pi}\left\{\int_0^c \left[\frac{1}{s-r}h_1(r,s) + h_3(r,s)\right]f_1(s)\,\mathrm{d}s \right.$$
$$\left. + \int_a^b \left[\frac{1}{s-r}h_1(r,s) + h_3(r,s)\right]f_2(s)\,\mathrm{d}s\right\} = p_e - Ph_4(r), \qquad a < r < b, \tag{38a,b}$$

$$\int_a^b f_2(r)\,\mathrm{d}r = 0, \tag{39}$$

where

$$f_1(r) = \frac{\partial}{\partial r}w(r,0), \qquad 0 \le r < c,$$

$$f_2(r) = \frac{\partial}{\partial r}w(r,0), \qquad a < r < b,$$

$$h_3(r,s) = \pi\int_0^\infty E_1(\alpha)J_0(\alpha r)J_1(\alpha s)\alpha\,\mathrm{d}\alpha,$$

$$h_4(r) = r_0\int_0^\infty E_2(\alpha)J_0(\alpha r_0)J_0(\alpha r)\alpha\,\mathrm{d}\alpha. \tag{40a–d}$$

The functions E_1, E_2, h_1 and h_2 are defined by (19) and (21). The indexes of both singular integral equations (38a) and (38b) are -1. Thus, the two consistency conditions and the single-valuedness condition (39) provide three equations to determine the unknown constants a, b and c. In practice the problem may easily be solved numerically by normalizing the intervals $(0, c)$ and (a, b) and by using the technique mentioned in Section 4.

If r_0 is increased further, due to the loads $p_0 + \rho g h = p_e$, the center portion of the circular separation region would collapse and one would have two ring-shaped separation regions, $d < r < c$ and $a < r < b$ where $0 < d < c < r_0 < a < b$. In this case defining

$$f_1(r) = \frac{\partial}{\partial r}w(r,0), \qquad d < r < c,$$

$$f_2(r) = \frac{\partial}{\partial r}w(r,0), \qquad a < r < b, \tag{41a,b}$$

the integral equations become

$$\int_d^c h(r,s)f_1(s)\,ds + \int_a^b h(r,s)f_2(s)\,ds = p_e - Ph_4(r), \qquad d<r<c,$$
$$\int_a^b h(r,s)f_2(s)\,ds + \int_d^c h(r,s)f_1(s)\,ds = p_e - Ph_4(r), \qquad a<r<b, \tag{42a,b}$$

$$h(r,s) = \frac{1}{\pi}\frac{4\mu}{1+\kappa}\left[\frac{h_1(r,s)}{s-r} + h_3(r,s)\right]. \tag{43}$$

Note that in (42) only the first terms on the left hand side have kernels with Cauchy type singularity. Here the unknown constants a–d are determined from two consistency conditions of the integral equations and the following two single-valuedness conditions

$$\int_d^c f_1(r)\,dr = 0, \qquad \int_a^b f_2(r)\,dr = 0. \tag{44a,b}$$

7. RESULTS

For relatively small values of r_0/h and $\gamma > \gamma_{cr} > 0$ (i.e. for lifting force) some of the calculated results corresponding to a circular separation region (i.e. for $a = 0$) are given in Figs. 4–6. Figure 4 shows the radius of the separation region b as a function of γ for fixed values of r_0/h. The figure also shows the parabola obtained from the plate theory as given by (35). As seen from the figure, the plate theory predicts a separation area which is considerably greater than that given by the elasticity solution. One may also note that according to the plate theory continuous contact (i.e. $b = 0$) is possible only for $\gamma = 0$ (or, for $P = 0$) which is clearly very unrealistic. However, as γ increases, the relative differences between the results calculated from the two procedures would decrease and the results obtained from the plate theory could become acceptable. Similar results were found in [11] for the plane problem.

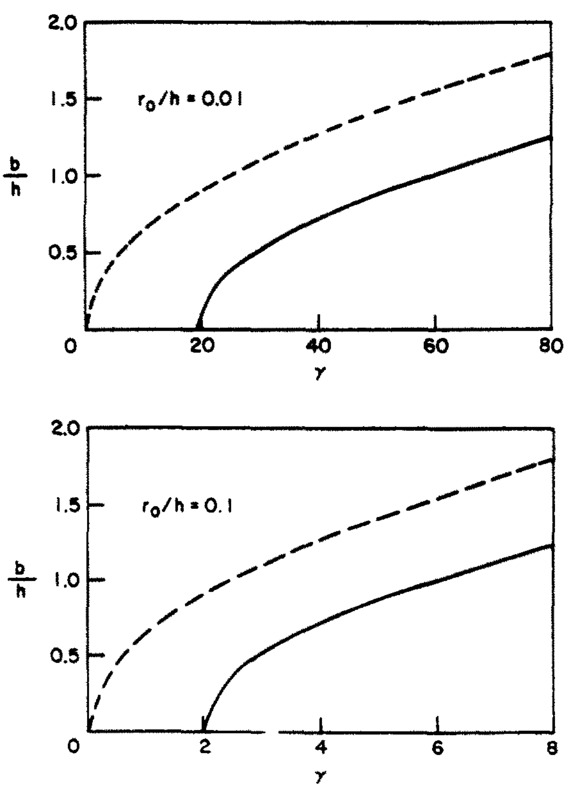

Fig. 4. Variation of the radius of the circular separation area as a function of loading ratio $\gamma = P/h(\rho gh + p_0)$. (Full line: elasticity solution, dashed line: plate approximation).

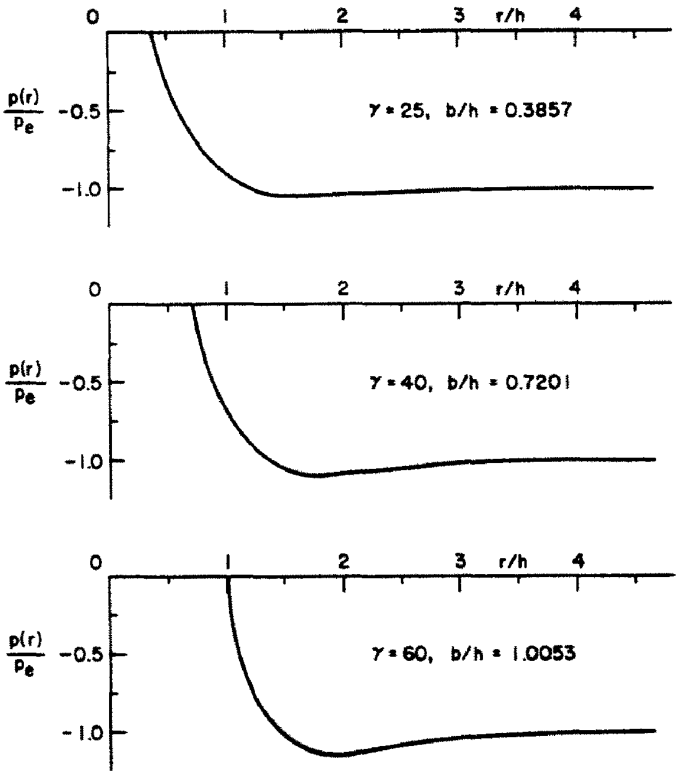

Fig. 5. Distribution of the contact stress $\sigma_{zz}(r, 0) = p(r)$ for various values of γ and for $r_0/h = 0.01$ in the case of circular separation area ($p_e = \rho g h + p_0$).

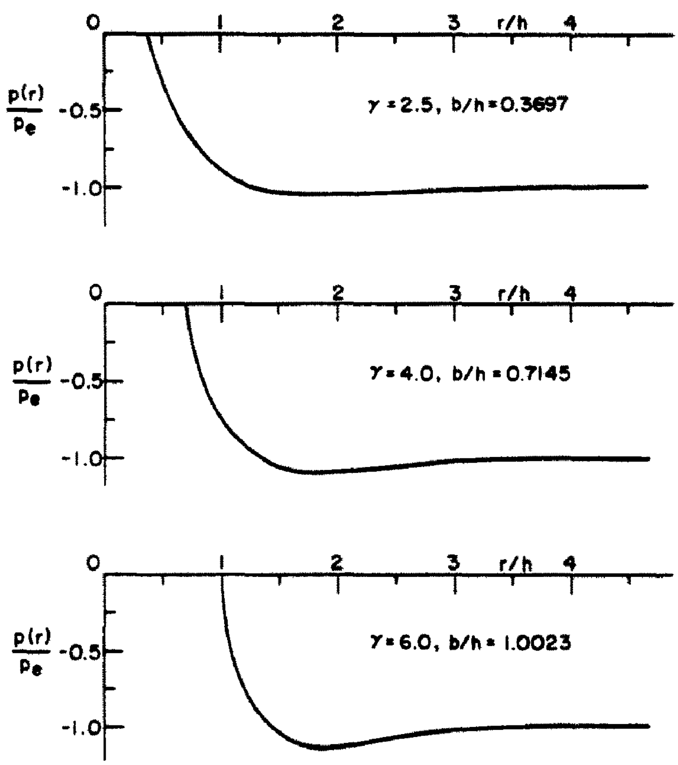

Fig. 6. Distribution of the contact stress $\sigma_{zz}(r, 0) = p(r)$ for various values of γ and for $r_0/h = 0.1$ in the case of circular separation area ($p_e = \rho g h + p_0$).

Figures 5 and 6 show the distribution of the contact stress $\sigma_{zz}(r, 0) = p(r)$, $r > b$ for fixed values of the load ratio $\gamma = P/hp_e$. The calculated values of the corresponding separation radius are also shown in the figures. The contact stress obtained from the classical plate theory is given by (36), namely

$$p(r) = \begin{cases} 0, & 0 \leq r < b, \\ -\left(p_e + \dfrac{r_0 P}{b}\right), & r = b, \\ -p_e, & b < r < \infty. \end{cases} \tag{45}$$

This tendency of concentration of the contact pressure around $r = b$ may also be observed in Figs. 5 and 6.

The results for the ring-shaped separation region (i.e. for $0 < a < b$), again for the case of lifting force, are given in Figs. 7–9. Figure 7 shows, for two fixed values of load radius r_0/h and for $\gamma > \gamma_{cr}$, the radii of the separation area, a and b. The figure also shows the transition value of γ at which the ring-shaped separation area becomes circular. The value of the corresponding radius b of this circle is also indicated in the figure.

Some sample results for the contact stress distribution $\sigma_{zz}(r, 0) = p(r)$ are shown in Figs. 8 and 9 for fixed values of loading radius r_0/h and the load ratio $\gamma = P/hp_e$. Again, the figures also show the corresponding contact radii a and b.

Figures 10–12 show the results for the compressive force. For values of $r_0 = 0.01, 0.1$ and 1.0 the figures show the values of the load ratio $\gamma_{cr} = P_{cr}/hp_e$ and the radius r_{cr} at which the separation would start. Figures 10 and 11 also show a sample result for which $|\gamma| > |\gamma_{cr}|$ and the corresponding radii a and b of the separation region. In order to include the entire pressure distribution and to include sufficient details for $|p(r)/p_e| < 1$, in figures different scales have been used for $p > p_e$ and for $p < p_e$. Note that for very small values of r_0/h the maximum pressure is at $r = 0$. However, as r_0 increases $r = 0$ becomes the point of a local minimum for pressure (see Fig. 12) indicating that for certain combinations of r_0 and γ there may be an additional separation region around $r = 0$.

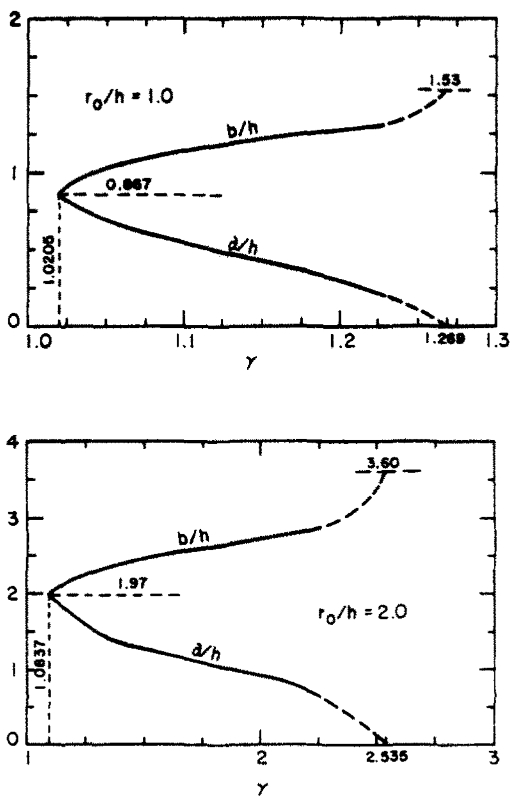

Fig. 7. The radii a and b of the ring-shaped separation area.

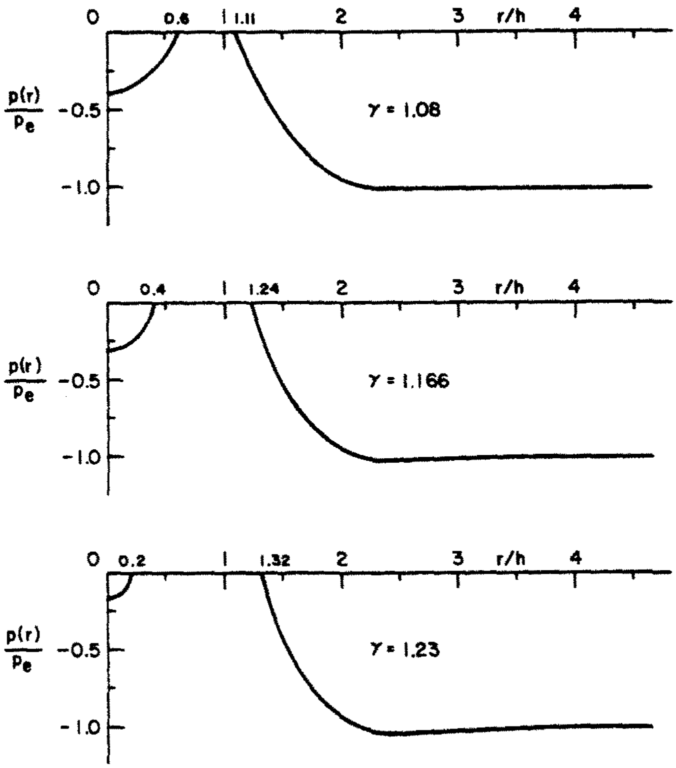

Fig. 8. Distribution of the contact stress $\sigma_{zz}(r, 0) = p(r)$ in the case of ring-shaped separation area for $r_0/h = 1$, $(p_e = \rho g h + p_0)$.

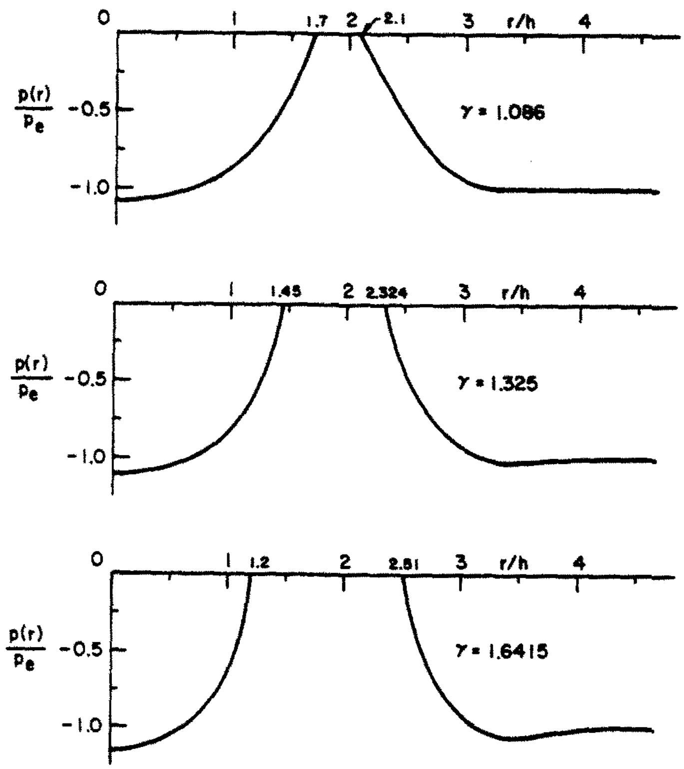

Fig. 9. Distribution of the contact stress $\sigma_{zz}(r, 0) = p(r)$ in the case of ring-shaped separation area for $r_0/h = 2$, $(p_e = \rho g h + p_0)$.

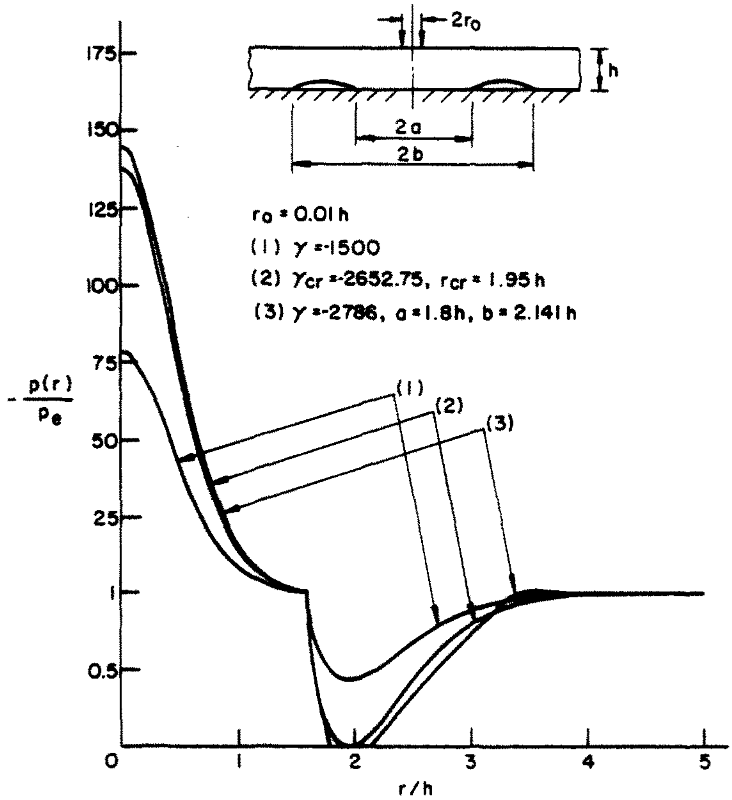

Fig. 10. Distribution of contact stress for compressive loading, $\gamma = P/(hp_e)$, $p_e = p_0 + \rho g h$, $r_0 = 0.01$ h.

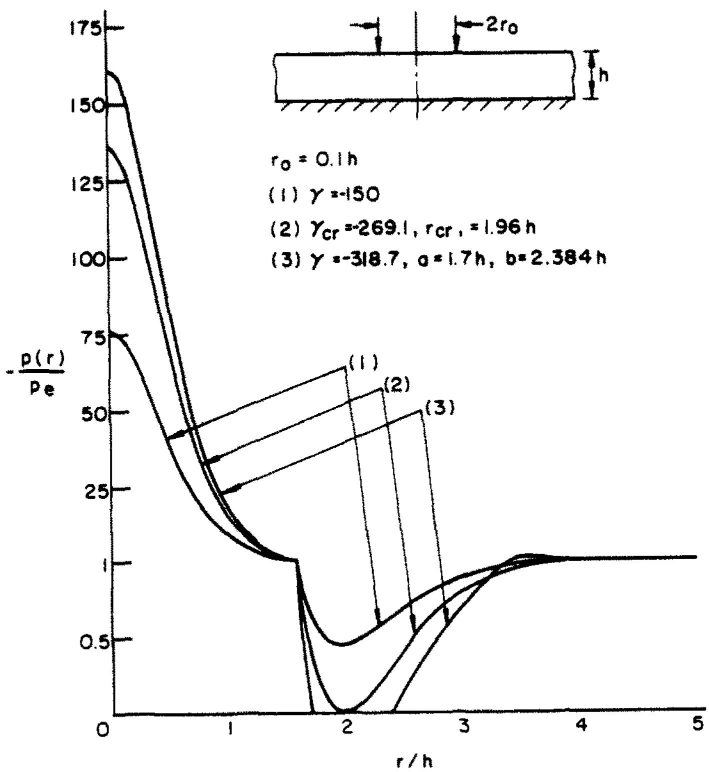

Fig. 11. Distribution of contact stress for compressive loading, $\gamma = P/(hp_e)$, $p_e = p_0 + \rho g h$, $r_0 = 0.1$ h.

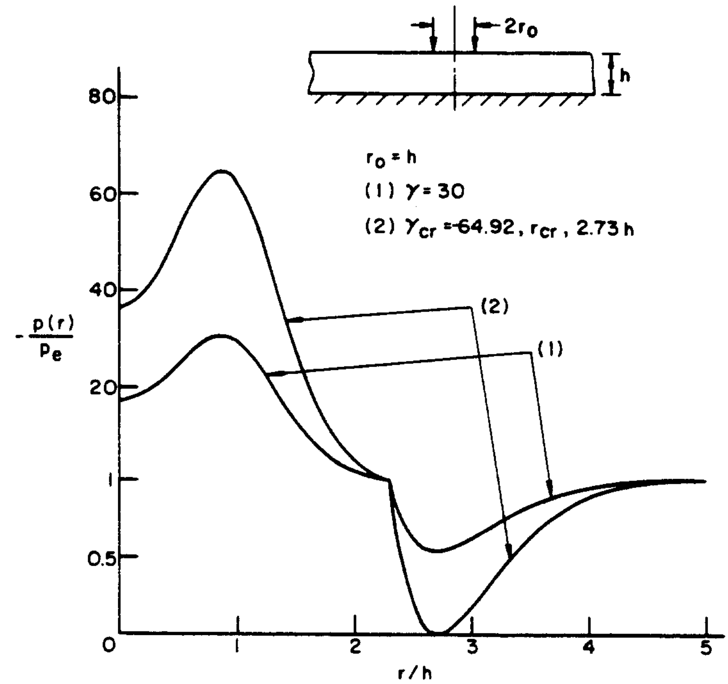

Fig. 12. Distribution of contact stress for compressive loading, $\gamma = P/(hp_e)$, $p_e = p_0 + \rho gh$, $r_0 = h$.

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